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DETERMINATION OF THE TOPOLOGICAL STRUCTURE OF AN ORBIFOLD BY ITS GROUP OF ORBIFOLD DIFFEOMORPHISMS

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Given a topological space X with some geometric structure (including topological structures, differentiable structures, symplectic structures and contact structures) and the group of transformations that preserve these structures (the group of homeomorphisms, diffeomorphisms, symplectic diffeomorphisms and contact diffeomorphisms), one can ask whether these groups of structure preserving transformations determine the corresponding structures. The topological case has been studied by Gerstenhaber [?], Fine and Schweigert [?], Rubin [?], Wechsler [?], and Whittaker [?]. The differentiable case has been studied by Banyaga [?], Filipkiewicz [?], Rubin [?] and Rybicki [?]. The symplectic and contact cases have been studied by Banyaga [?]. Rubin [?] has also studied many other variants of this question including the PL, Lipschitz and quasiconformal cases. A key ingredient in our proof in the orbifold case will be a theorem of Rubin:

Theorem (Rubin). *Let $X_i, (i = 1, 2)$ be locally compact Hausdorff spaces and G_i subgroups of the group of homeomorphisms of X_i such that for every open set $T \subset X_i$ and $x \in T$ the set $\{g(x) \mid g \in G_i \text{ and } g\,|_{(X_i - T)} = \mathrm{Id}\}$ is somewhere dense. Then if $\Phi : G_1 \to G_2$ is a group isomorphism, then there is a homeomorphism h between X_1 and X_2 such that for every $g \in G_1$, $\Phi(g) = hgh^{-1}$.*

Recall that a subset S of a topological space X is called *somewhere dense* if the interior of its closure is nonempty. That is, $\mathrm{int}(\mathrm{cl}(S)) \neq \emptyset$. Our theorem is the following:

Theorem 1. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two compact, locally smooth orbifolds. Fix $r \geq 0$. Suppose that $\Phi : \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}_1) \to \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}_2)$ is a group isomorphism. Then Φ is induced by a homeomorphism $h : X_{\mathcal{O}_1} \to X_{\mathcal{O}_2}$. That is, $\Phi(f) = hfh^{-1}$ for all $f \in \mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O}_1)$. Furthermore, if $r > 0$, h is a C^r manifold diffeomorphism when restricted to the complement of the singular set of each stratum.*

Here, $\mathrm{Diff}^r_{\mathrm{Orb}}(\mathcal{O})$ denotes the C^r orbifold diffeomorphism group and $X_{\mathcal{O}}$ the underlying topological space of an orbifold $\mathcal{O}$. We review the definitions of these notions in the next few sections. The restriction in Theorem 1 to compact orbifolds cannot be removed as the following example shows.

Example 2. Let $\mathcal{O}_1 = (0, 1)$ and $\mathcal{O}_2 = [0, 1]$, the open and closed unit intervals. These orbifolds have the same homeomorphism group, but are clearly not homeomorphic spaces.

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In general, the homeomorphism h in Theorem 1 is not necessarily an orbifold homeomorphism. To see this, consider the following

Example 3. Let $\mathcal{O}_i, (i = 1, 2)$ be two so–called $\mathbb{Z}_{p_i}$–teardrops with $p_1 \neq p_2$. It is clear that the homeomorphism groups of $\mathcal{O}_i$ are each isomorphic to the subgroup of the homeomorphism group of the 2–sphere S^2 which fix the north pole. To see this, just observe that any homeomorphism of S^2 that fixes the north pole can be locally lifted to a p_i–fold covering of a neighborhood of the north pole. Note, however that the orbifolds themselves are not *orbifold* homeomorphic, even though their underlying spaces $X_{\mathcal{O}_i} = S^2$, are topologically homeomorphic.

If one considers two non–homeomorphic Riemannian manifolds with trivial isometry group one is easily convinced that if the automorphism group of a particular structure is not rich enough then the underlying topological structure cannot be determined at all. Before proceeding with the proof of our theorem we need to review some definitions involving the orbifold category.

1. Orbifold Preliminaries

2. Elementary Properties of Locally Smooth Orbifolds

3. Examples

4. Maps Between Orbifolds

5. Extending Orbifold Diffeomorphisms

6. Local Contractions

7. Proof Of Theorem 1

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