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ANOSOV-LIKE ACTIONS OF FINITELY GENERATED GROUPS

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Abstract. We present a generalization of one possible formulation of the Anosov conditions for diffeomorphisms to actions of a finitely generated group Γ on a smooth, closed manifold. The hypotheses necessary for the generalization require triviality of cohomolgy with coefficients in the continuous tangent vector fields in degrees 0 and 1 of the acting group Γ. Actions satisfying these conditions are defined to be Anosov-like and we show that a diffeomorphism of a smooth manifold M is Anosov if and only if the corresponding $\mathbb{Z}$ action is Anosov-like. Examples are constructed of Anosov-like actions of $\mathbb{Z}^n$ on nilmanifolds, and it is shown that the standard action of $SL_n(\mathbb{Z})$ on the n-torus via linear transformations is Anosov-like. This generalizes the stability properties of Anosov diffeomorphisms to more general group actions.

1. Background

Let Γ be a finitely generated group. Given a topological group G, the set of representations of Γ in G will be denoted by $R(\Gamma, G)$. This can be given the structure of a topological space by taking the topology to be the compact-open topology. The compact-open topology may also be shown to be the topology of uniform convergence on finite sets of generators of Γ. Given $\phi \in R(\Gamma, G)$, it is natural to try to determine the local structure of $R(\Gamma, G)$ in a neighborhood of ϕ. For the case of G a Lie group with Lie algebra $\mathfrak{g}$, a classical result of Weil [22] states that if $H^1(\Gamma, \mathfrak{g}_{ad\phi}) = 0$, then all representations ψ of Γ in a sufficiently small neighborhood of ϕ in $R(\Gamma, G)$ are conjugate to ϕ via an element of G. The subscript $ad\phi$ refers to the action of Γ on $\mathfrak{g}$ via the composition of ϕ with the adjoint action ad of G on $\mathfrak{g}$. A representation ϕ of Γ with this property is said to be *locally rigid*. The first cohomology group $H^1(\Gamma, \mathfrak{g}_{ad\phi})$ may be computed using techniques such as those developed by Matsushima and Murakami [16].

Recently, there has been interest in extending this body of theory to a broader class of topological groups. Of interest is the diffeomorphism group $\mathrm{Diff}^\infty(M)$ of a smooth closed manifold M. The analog of the Lie algebra for $\mathrm{Diff}^\infty(M)$ is $D^\infty(M)$, the space of C^∞ tangent vector fields on M and the analog of $ad\phi$ is ϕ_*. A representation of Γ satisfying $H^1(\Gamma, D^\infty(M)_{\phi_*}) = 0$ is said to be *infinitesimally rigid*. Zimmer [23] conjectured that, as for Lie groups, infintesimal rigidity implies local rigidity.

In [6], a partial verification of this conjecture was obtained for actions of a finitely generated group on a smooth, closed (i.e. compact, without boundary) manifold M. The main aim of this paper is to extend the results presented there to the

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topological case. Recall that an action ϕ of a group Γ is said to be C^r *locally rigid*, $r = 0, \ldots \infty$ if there is a neighborhood of ϕ in $Hom(\Gamma, \mathrm{Diff}^r(M))$ (given the compact-open topology) so that any action ψ in this neighborhood is conjugate to ϕ, the case $r = 0$ being the case of continuous or *topological* rigidity. In the case of actions of the additive group of the integers (a dynamical system generated by a diffeomorphism $f \in \mathrm{Diff}(M)$, these conditions are equivalent to the action being Anosov. With this in mind, we define a Γ action, ϕ to be *Anosov-like* if it satisfies $H^1(\Gamma, D^0(M)) = H^0(\Gamma, D^0(M)) = 0$. The major theorem in this regard is the following:

Theorem A. *With Γ, M and ϕ as above, if $H^1(\Gamma, D^0(M)) = H^0(\Gamma, D^0(M)) = 0$, then any C^1 Γ action sufficiently close to ϕ on a finite set of generators is topologically conjugate to ϕ. Moreover, the condition $H^1(\Gamma, D^0(M)) = H^0(\Gamma, D^0(M)) = 0$ is satisfied by all Γ actions in a neighbourhood of ϕ in $Act^1(\Gamma, M)$.*

The first example of a Γ action satisfying these criteria is the $\mathbb{Z}$ action on a smooth, closed manifold induced by the action of an Anosov diffeomorphism. In fact, a theorem of Mather [15] shows that a $\mathbb{Z}$ action satisfies $H^1(\mathbb{Z}, D^0(M)) = H^0(\mathbb{Z}, D^0(M)) = 0$ iff the image of a generator of $\mathbb{Z}$ is an Anosov diffeomorphism. By Theorem A, Anosov-like actions are open in the space of Γ actions and so Theorem A recaptures all the relevant topological stability results for Anosov diffeomorphisms.

Further examples may be generated via the following 2 theorems that serve to verify that many actions are in fact Anosov-like.

Theorem B. *Let Γ_i, $i = 1, 2$ be finitely generated groups and $\phi_i \in \mathrm{Diff}^1(\Gamma_i, M_i)$ Anosov-like actions on smooth, closed manifolds M_i. Then the product action of $\Gamma_1 \times \Gamma_2$ on $M_1 \times M_2$ is also Anosov-like.*

To get examples of Anosov-like actions that do not come from product actions, we consider actions of lattice subgroups of Lie groups. The ones that we will consider come from groups satisfying the *Strong Vanishing Condition* or *SVC*. The definition of this appears in [12]

Theorem C. *Let $\phi \in Act^r(\Gamma, M)$ be an action of a finitely generated group Γ with the SVC satisfying:*

1. *The periodic points of ϕ, $\mathrm{Per}(\phi)$ are dense in M*
2. *The action contains an Anosov diffeomorphism.*

then the action ϕ is Anosov-like.

For the sake of clarity, the conjecture that provides the motivation for this work is as follows. Zimmer [23] conjectured that if a finitely generated group Γ acts smoothly via an action ϕ on a smooth, closed manifold M so that $H^1(\Gamma, D^\infty(M)_{\phi_*}) = 0$, then the action ϕ is *locally rigid*, i.e. any action ψ of Γ sufficiently close to ϕ is C^∞ conjugate to ϕ. This would be a generalization of Weil's [22] local rigidity result for discrete subgroups of Lie groups to the infinte dimensional Lie group $\mathrm{Diff}^\infty(M)$. A partial verification of Zimmer's conjecture follows from the following

Theorem D. *[Theorem 3.5 of [6]] Let Γ be a finitely generated group, M a C^2 Banach manifold modelled on a Banach space V, $\phi \in Act^1(\Gamma, M)$ an action of Γ on M satisfying*

1. *ϕ fixes a point $p \in M$*

(2) *if* $\Phi \in R(\Gamma, GL(V))$ *is the linear action obtained by linearization of* ϕ *at* p, *then* $H^1(\Gamma, V) = 0$, *where* V *is a* Γ *module via* Φ

Then, p *is a stable fixed point, i.e. for all neighborhoods* U *of* p *there is a neighborhood* W *of* ϕ *in* $Act^1(\Gamma, V)$ *so that all actions* $\psi \in W$ *have fixed points in* U.

A consequence of this is the following Corollary

Corollary 1. *Let* Φ *be a* C^r, *(for* $r \geq 2$*) action of a finitely generated group* Γ *on a smooth, closed manifold* M *satisfying*

$$H^1(\Gamma, D^{r-1}(M)_{\Phi_*}) = 0$$

Then there is a neighborhood $\mathcal{U}$ *of* Φ *in the space of such actions so that for each action* $\Psi \in \mathcal{U}$, *there is a* C^{r-1} *diffeomorphism* f *of* M, *so that*

$$f \circ \Phi = \Psi \circ f$$

The results conatined in this paper generalize the stability properties of Anosov diffeomorphisms using this cohomological approach. Verifying these criteria in practice requires a good knowlege of the dynamical structure of the group actions - which are typically ergodic and contain Anosov diffeomorphisms. An interesting question is to what extent the cohomological stability criteria determine the dynamics - or at least what properties of the dynamics of the group action can be recoverd from knowing the triviality of the cohomology.

It is possible to generalize these even further (e.g. to relativize these results to the case of a closed invariant set) or to more general spaces such as *orbifolds* (for an example see Borzellino and Brunsden [4]).

2. Definitions

2.1. **Banach Manifolds and Topological Vector Spaces.** All vector spaces are assumed real with any exceptions explicitly indicated. Sources for material on Banach manifolds are Ebin and Marsden [9], Eells [8], Lang [13] or Omori [17]. Let V be a Banach space (i.e. a metrizable, locally convex, topological vector space with metric gvien by a *norm* $|\ |$ which is positively homogeneous with respect to real number multiplication and satisfies the triangle inequality). A paracompact, Hausdorff space M is a *Banach manifold modelled on* V, $r \geq 0$ if there is a collection of pairs (called *charts*) $(U, \phi)_{U \in \mathcal{U}}$ where $\mathcal{U}$ is an open covering of M and $\phi_U : U \to V$ is a homeomorphism of U to an open subset of V. M is a C^r *Banach manifold* if for all charts (U, ϕ) and (V, ψ) with $U \cap V \neq \emptyset$, the mapping $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is a C^r mapping. As in the finite dimensional case the *tangent bundle* TM of a C^r Banach manifold M is defined to be the C^{r-1} Banach manifold modelled on the Banach space $V \times V$ with charts $(U \times V, \phi \times d\phi)_{U \in \mathcal{U}}$. The notions of a C^r mapping between such spaces, its derivative and *k-jet* are as in the finite dimensional case and may be found in the previously mentioned references as well as in Deimling [7]. Unlike the finite dimensional case, not all C^r Banach manifolds necessarily have a compatible C^∞ structure - though all the ones discussed in this paper do so.

With the above definitions of a Banach manifold, it is a classical result that the group $\mathrm{Diff}^r(M)$ of C^r diffeomorphisms of a smooth, closed manifold M, is itself a C^∞ Banach manifold with model Banach space, the space $D^r(M)$ of C^r tangent vector fields to M equipped with the topology of uniform convergence of derivatives of order $\leq r$ on compact subsets of M. The proof may be found in Ebin and

Marsden [9], Omori [17], Eells [8] or Abraham [1]. With multiplication in $\mathrm{Diff}^r(M)$ given by composition, $\mathrm{Diff}^r(M)$ becomes a topological group that is also a Banach manifold (not however, a Banach Lie group as right multiplication is not smooth). Another example of an infinite dimensional topological group having the structure of a Banach manifold is $C^r(M,G)$, the space of C^r maps from a smooth, closed manifold M to a Lie group G. Again the topology is that of uniform convergence of all derivatives of order $\leq r$ on compact subsets of M and with this topology, it is a Banach Lie group. While $\mathrm{Diff}^r(M)$ is a manifold and a topological group, multiplication (given by composition) is not smooth, though it is Lipschitz and right translation by a fixed element of $\mathrm{Diff}^r(M)$ is smooth. This is also true for $C^0(M,M)$ when M is a smooth manifold, with model space $D^0(M)$ for $C^0(M,M)$. For a proof, see Banyaga [2].

2.2. **The spaces $R(\Gamma,G)$ and $\mathrm{Act}^r(\Gamma,M)$.** Let G be a topological group and Γ a finitely generated group. Fix a presentation

$$1 \longrightarrow R \longrightarrow F \longrightarrow \Gamma \longrightarrow 1$$

of Γ, where F is free on a finite subset S of Γ and R is the kernel of the epimorphism mapping F to Γ. Let S consist of the elements $\gamma_1,\ldots,\gamma_n$ and R be freely generated (see Robinson [20]) by $r_1, r_2, r_3, \ldots$ where $r_i \in F$ for $i = 1,\ldots n$. If $\psi : \Gamma \to G$ is a homomorphism, then clearly ψ need only be specified on generators. In particular, since ψ is a homomorphism,

$$\psi(r(\gamma_1,\ldots,\gamma_n)) = r(\psi(\gamma_1),\ldots,\psi(\gamma_n))$$

for all $r \in R$. Suppose that $\psi(\gamma_i) = g_i$ for $i = 1,\ldots,n$, then

$$r(g_1,\ldots,g_n) = 1_G$$

for all $r \in R$. Conversely, given $g_1,\ldots,g_n \in G^n$ that satisfying

$$r(g_1,\ldots,g_n) = 1_G$$

for all $r \in R$, assigning $\psi(\gamma_i) = g_i$ for i $= 1,\ldots,n$ defines a homomorphism $\psi : \Gamma \to G$. In this fashion, for a presentation of Γ as described above, the set $\mathrm{Hom}(\Gamma,G)$, of homomorphisms from Γ to G may be regarded as the subset

$$\{(g_1,\ldots,g_n) \in G^n | r(g_1,\ldots,g_n) = 1_G \text{ for all } r \in R\}$$

of G^n. G^n with the product topology thus induces a topology on $\mathrm{Hom}(\Gamma,G)$. It is straightforward to check that this topology is independent of the presentation and coincides with the compact-open topology on $\mathrm{Hom}(\Gamma,G)$ when Γ is given the discrete topology. We define $R(\Gamma,G)$ to be $\mathrm{Hom}(\Gamma,G)$ equipped with this topology. If M is a smooth closed manifold, let $\mathrm{Act}^r(\Gamma,M)$ be the space $R(\Gamma,\mathrm{Diff}^r(M))$, where $\mathrm{Diff}^r(M)$ is given the topology of uniform convergence of derivatives of order $\leq r$ on compact subsets of M.

Finally, we define the *Fox derivative*. Given a presentation of Γ,

$$1 \longrightarrow R \longrightarrow F \longrightarrow \Gamma \longrightarrow 1$$

where F is free on $S \subset \Gamma$ as above, $\psi \in Hom(\Gamma, Aut(A))$ and $(a_1,\ldots) \in A^{|S|\cdot}$, the *Fox derivative* of $w \in F$ is defined as follows. Composition of ψ with the quotient homomorphism $F \longrightarrow \Gamma$ gives a homomorphism which will also be denoted by ϕ. Let $S = \{\gamma_i\}_{i\in\mathcal{I}}$, given $\{a_i\}_{i\in\mathcal{I}}$ where $a_i \in A$ there is a $\hat{\phi} \in Hom(F,\mathrm{Aff}(A))$ with $\hat{\phi}(\gamma_i)(a) = \phi(\gamma_i)(a) + a_i$ for all $i \in \mathcal{I}$. The Fox derivative of $w \in F$ written

$dw_\phi(a_1,\ldots)$ is then defined to be $\hat{\phi}(w(\gamma_1,\ldots))(0)$. The Fox derivative satisfies $d(w_1w_2)_\phi(a_1,\ldots) = d(w_1)_\phi(a_1,\ldots) + \phi(w_1(\gamma_1,\ldots))(d(w_2)_\phi(a_1,\ldots))$ for all words $w_1, w_2 \in F$.

2.3. **Cohomology.** For the purposes of this paper, a *complex* $C_\bullet$ is a family of pairs $\{V_i, \partial_i\}_{i\in\mathbb{Z}}$, indexed by the integers, where each V_i is a vector space, and

$$\partial_i : V_i \to V_{i-1}$$

is a family of linear transformations satisfying $\partial_{i-1}\partial_i =$ for all i. The *homology groups* of a complex $C_\bullet$ are the quotient spaces,

$$H_i(C_\bullet) = \ker(\partial_i)/\operatorname{im}(\partial_{i+1})$$

A complex for which $V_i = \{0\}$ for $i < 0$ is called a *chain complex* and one for which $V_i = \{0\}$ for $i > 0$ is usually called a *co-chain complex* and is usually written with superscripts and increasing indices, i.e. $V^{-i} = V_i$ for $i < 0$ and $\partial^{-i} = \partial_i$ and the homology groups are usually called *cohomology* and are written as

$$H^i(C^\bullet) = \ker(\partial^i)/\operatorname{im}(\partial^{i-1})$$

For a ring R (which in the cases considered here will usually be the group ring $\mathbb{Z}[\Gamma]$ of our acting group Γ, a projective module M is an R module having the property that for any R modules S and T a diagram

$$\begin{array}{ccc} & & M \\ & & \big\downarrow{\scriptstyle p} \\ S & \xrightarrow{f} & T \end{array}$$

where the arrows represent R homomorphisms, may be uniquely completed to a commutative diagram

$$\begin{array}{ccc} & & M \\ & {\scriptstyle q}\swarrow & \big\downarrow{\scriptstyle p} \\ S & \xrightarrow{f} & T \end{array}$$

The cohomology groups occurring in this paper arise from (co-)chain complexes built out of the coefficient group (always a Banach space in this context) and a projective $\mathbb{Z}[\Gamma]$ resolution of $\mathbb{Z}$.

$$C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} \cdots$$

let $Z^i(C^\bullet) = \ker(\partial^i)$ denote the cocoycles in degree i and $B^i(C^\bullet) = \operatorname{im}(\partial^{i-1})$ the coboundaries in degree i. Usually, a cochain complex will simply be referred to as $C^\bullet$ with the coboundary operators $\partial^\bullet$ being understood. The complexes arise by applying the functor $\operatorname{Hom}_\Gamma(\ \ , V)$ to a projective $\mathbb{Z}[\Gamma]$ resolution (an *exact* complex i.e $\ker(\partial_n) = \operatorname{im}(\partial_{n+1})$ for all n)

$$\cdots \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} \mathbb{Z} \xrightarrow{\partial} 0$$

of $\mathbb{Z}$.

A cochain complex computing the cohomology of a finitely generated group Γ may obtained from the *Gruenberg Resolution* corresponding to a presentation

$$1 \longrightarrow R \longrightarrow F \longrightarrow \Gamma \longrightarrow 1$$

of Γ where F is a free group on a finite set of generators $S \subset \Gamma$. The Gruenberg resolution associates to such a presentation a free $\mathbb{Z}[\Gamma]$ resolution $P_\bullet$

$$\cdots \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} \mathbb{Z} \xrightarrow{\partial} 0$$

of $\mathbb{Z}$, where the P_is are countably generated if Γ is. The modules P_n are defined via

$$P_{2n+1} = I_F \overline{I}_R^n / I_F \overline{I}_R^{n+1} \tag{2.1}$$

$$P_{2n} = \overline{I}_R^n / \overline{I}_R^{n+1} \tag{2.2}$$

where I_G is the kernel of the G map $\mathbb{Z}[G] \to \mathbb{Z}$ and $\overline{I}_N = \mathbb{Z}[G]I_N$ is the kernel of the homomorphism $\mathbb{Z}[G] \to \mathbb{Z}[G/N]$ for any normal subgroup N of a group G. The coboundary maps are defined by $\partial^n(f)(x) = f(\partial_n(x))$ for $f \in \mathrm{Hom}_\Gamma(P_n, V)$. A full treatment of the construction of these can be found in Robinson, [20] or Hilton and Stambach [11]. We should note that for a free group F_n on n generators, we have

$$H^i(F_n, V) = 0$$

for all $i \geq 2$. In particular, this implies that $H^i(\mathbb{Z}, V) = 0$ for all $i \geq 2$ since the additive group of integers is free (on 1 generator).

Although we make no use of this, a short exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

of $\mathbb{Z}[\Gamma]$ modules (i.e. $W = V/W$ as $\mathbb{Z}[\Gamma]$-modules), induces a long exact sequence of cohomology groups:

$$\cdots \longrightarrow H^i(\Gamma, U) \longrightarrow H^i(\Gamma, V) \longrightarrow H^i(\Gamma, W) \xrightarrow{\delta^i} H^{i+1}(\Gamma, U) \longrightarrow \cdots$$

Such a situation would occur in dynamically relevant situations where there is a Γ invariant closed subset $C \subset M$, let $V = D^0(M)$, $W = D^0(M)|_C$ which forces $U = \{v \in D^0(M) | v|_C = 0\}$.

The low dimensional cohomology groups can be given the following interpretations. The 0-th cohomology group may be identified as

$$H^0(\Gamma, V) = \{v \in A | v = \varphi(\gamma)v \text{ for all } \gamma \in \Gamma\}.$$

The group of 1-cocycles may be interpreted to be

$$Z^1(\Gamma, V) = \{z : \Gamma \to V | v \mapsto \varphi(\gamma)v + z(\gamma) \in \mathrm{Hom}(\Gamma, \mathrm{Aff}(V))\}.$$

A function $z : \Gamma \to V$ satisfying this condition is called a *crossed homomorphism*. The subgroup consisting of the coboundaries may be interpreted as the set of all cocycles for which there is an $a \in A$ which is fixed by this affine action. That is, $\varphi(\gamma)a + z(\gamma) = a$ for all $\gamma \in \Gamma$. They are referred to as *principal crossed homomorphisms*. The first cohomology group with coefficients in A is then the quotient of the group of crossed homomorphisms by the group of principal, crossed homomorphisms. In other words,

$$\begin{aligned} H^1(\Gamma, V) &= Z^1(\Gamma, V)/B^1(\Gamma, V) \\ &= \{z : \Gamma \to V | z(\gamma\delta) = z(\gamma) + \varphi(\gamma)z(\delta)\}/\sim \end{aligned}$$

where the equivalence relation $\sim$ is defined by

$$z(\gamma) \sim w(\gamma) \Leftrightarrow z(\gamma) - w(\gamma) = v - \varphi(\gamma)v$$

As many of the the examples that follow rely on this condition (as does the statement of 1), it is appropriate to include the following

Definition 2. A finitely generated group Γ satisfies the *Strong Vanishing Condition* if

$$H^1(\Gamma', \mathbb{R}^n_\phi) = 0$$

for all subgroups Γ' of finite index in Γ and all representations $\phi \in R(\Gamma', GL_n(\mathbb{R}))$.

The main source of groups satisfying the SVC are lattice subgroups of semisimple Lie groups. For completeness, we recall the definition here.

Definition 3. A *lattice* is a discrete subgroup $\Gamma \subset G$ of a Lie group G such that the quotient G/Γ has finite volume.

The prototypical example of a lattice is of course $SL_n(\mathbb{Z})$ for $n > 3$. The main theorem that connects lattices to the SVC, due to Margulis [14] is the following.

Theorem 4. *Let $\Gamma \subset G$ be an irreducible lattice in a connected semisimple algebraic $\mathbb{R}$-group G. Assume further that the $\mathbb{R}$-split rank of each factor of G is at least 2 and that $G^0_{\mathbb{R}}$ has no compact factors. Then Γ satisfies the SVC.*

This will be used to construct our main example of non-trivial Anosov-like actions of finitely generated groups.

3. Examples

Example 5. Given $f \in \mathrm{Diff}^r(M)$ let $\phi_f \in \mathrm{Act}^r(\mathbb{Z}, M)$ be the action of $\mathbb{Z}$ on M determined by sending $1 \in \mathbb{Z}$ to f. This clearly induces a bijection between $\mathrm{Diff}^r(M)$ and $Act^r(\mathbb{Z}, M)$ (which is in fact a diffeomorphism of Banach manifolds - or Fréchet manifolds if $r = \infty$).

Theorem 6. $f \in \mathrm{Diff}^1(M)$ *is an Anosov diffeomorphism iff* $H^0(\mathbb{Z}, D^0(M)_{\phi_f}) = H^1(\mathbb{Z}, D^0(M)_{\phi_f}) = 0$.

Proof. By Theorem of Mather [15], f is Anosov if and only if $Id - f_* : D^0(M) \to D^0(M)$ is an isomorphism. Since

$$H^0(\mathbb{Z}, D^0(M)_{\phi_f}) = \ker(Id - f_*)$$

and

$$H^1(\mathbb{Z}, D^0(M)_{\phi_f} = \mathrm{coker}(Id - f_*)$$

this is equivalent to the conditions $H^0(\mathbb{Z}, D^0(M)_{\phi_f}) = H^1(\mathbb{Z}, D^0(M)_{\phi_f}) = 0$. □

Example 7. Let $\phi \in \mathrm{Act}^r(\Gamma, M)$ and $\Gamma' \subset \Gamma$ be a subgroup of finite index. Then ϕ is an Anosov-like Γ action if and only if $\phi|_{\Gamma'}$ is Anosov-like. Thus extensions of Anosov-like $\mathbb{Z}$-actions by a finite group are Anosov-like. Note that since $H^0(\Gamma, D^0(M))$ tends to be rather large if Γ is finite (simply average any $V \in D^0$

over Γ to obtain an invariant vector field), actions of finite groups are never Anosov-like. However, if Γ has an Anosov-like action ϕ on M, it may not be torsion free as Example 10 shows.

Example 8. Let $f_i \in \mathrm{Diff}^\infty(M_i)$ for $i = 1, 2$ be Anosov diffeomorphisms, then by Theorem B, the action of $\mathbb{Z}^2$ on $M_1 \times M_2$ defined by

$$(n_1, n_2) \times (x_1, x_2) \to (f_1^{n_1}(x_1), f_2^{n_2}(x_2))$$

is an Anosov-like action of $\mathbb{Z}^2$ on $M_1 \times M_2$.

Example 9. Let $f \in \mathrm{Diff}^\infty(\mathbb{T}^2)$ be an Anosov diffeomorphism of the 2-torus $\mathbb{T}^2$. Then a trivial inductive argument using 1 and 1 shows that the action of $\mathbb{Z}^n$ on T^{2n} defined by

$$(m_1, \ldots, m_n) \cdot (x_1, \ldots, x_n) = (f^{m_1}(x_1), \ldots, f^{m_n}(x_n))$$

where $x_i \in \mathbb{T}^2$ and $m_i \in \mathbb{Z}$, is Anosov-like. Generalizing this example to other nilmanifolds is essentially trivial. One can also vary the particular Anosov diffeomorphism acting on each factor to obtain other Anosov-like actions (as in the previous example).

Example 10. The standard action of $SL_n(\mathbb{Z})$ on $\mathbb{T}^n$ has dense periodic points (in fact these are the points that have rational coordinates with respect to the standard basis of $\mathbb{R}^n$.) By the above, the groups $H^1(\Gamma_p, T_p\mathbb{T}^n) = 0$ for each stabilizer subgroup $\Gamma_p \subset SL_n(\mathbb{Z})$ of a periodic point p. That $H^0(\Gamma_p, T_p\mathbb{T}^n) = 0$ for the same actions follows from the presence Anosov elements: i.e. there are $\gamma \in SL_n(\mathbb{Z})$ whose image in the standard action are Anosov diffeomorphisms. This reproduces the topological rigidity result of Hurder [12] for $SL_n(\mathbb{Z})$ but makes explicit the fact that rigidity in this case comes from the global condition of vanishing cohomology groups with coefficients in the continuous tangent vector fields. That the groups in question are lattices is required only to establish the triviality of the cohomology groups at the periodic points.

Example 11. Let G be a split, semisimple real Lie group with finite center and real rank ≥ 2; let Γ be an irreducible lattice in G, $\pi : G \to GL_n(\mathbb{R})$ a homomorphism and ϕ an action of Γ on a nilmanifold M by automorphisms associated with $\pi|_\Gamma$. If there is a $\gamma \in \Gamma$ so that $\phi(\gamma)$ is an Anosov diffeomorphism, then by [18] $H^1(\Gamma, D^0(M)) = 0$. Since $\phi(\gamma)$ is an Anosov diffeomorphism for some ϕ, it is clear that $H^0(\Gamma, D^0(M)) = 0$ and so the action ϕ is Anosov-like. In fact, this also follows from the Theorem C since Γ has the SVC and the periodic points are dense in M (by the results of [18]).

Example 12. Let $f_1, \ldots, f_n$ be Anosov diffeomorphisms of a manifold M and ϕ be the standard action of $SL_n(\mathbb{Z})$ on $\mathbb{T}^n$. Then the action of $SL_n(\mathbb{Z}) \times \mathbb{Z}^n$ on $\mathbb{T}^n \times M^n$ defined by $(A, m_1, m_2, \ldots m_n)(x, x_1, x_2, \ldots, x_n) = (A\cdot x, f^{m_1}(x_1), f_2^{m_2}(x_2), \ldots, f_n^{m_n}(x_n))$ is Anosov-like by combining the previous example and Theorem B. Here $A \in SL_n(\mathbb{Z})$ and $x_i \in M_i$ for $i = 1, \ldots, n$

4. Proof Of Theorem A

Before proceeding with the main part of the proof the following lemma will be particularly useful.

Lemma 13. *Let V be a Banach space $\Phi \in R(\Gamma, L(V))$ be a representation satisfying $H^1(\Gamma, V_\Phi) = 0$. Then there is a neighborhood $\mathcal{U}$ of Φ in $R(\Gamma, L(V))$ so that for all $\Psi \in \mathcal{U}$, $H^1(\Gamma, V_\Psi) = 0$. Moreover, if $H^0(\Gamma, V_\Phi) = 0$, then $H^0(\Gamma, V_\Psi) = 0$ for all $\Psi \in \mathcal{U}$.*

Proof. By Theorem 1 (Theorem 3.4 of [6]) for each neighborhood $W \subset V$ of 0, there is a neighborhood $\hat{\mathcal{U}}$ of Φ in $Act^1(\Gamma, V)$ so that each $\Psi \in \hat{\mathcal{U}}$ has a fixed point $p \in W$. Clearly $R(\Gamma, \mathrm{Aff}(V)) \subset \mathrm{Act}^1(\Gamma, V)$ in the obvious way. Hence any action in $\hat{\mathcal{U}} \cap R(\Gamma, \mathrm{Aff}(V))$ has a fixed point in W. This fixed point provides the solution of the coboundary equation for $z \in H^1(\Gamma, V_\Psi)$.

To prove the second part of the Lemma, by hypothesis If $v \in H^0(\Gamma, V_\Psi)$, where $||\Psi(\gamma) - \Phi(\gamma)|| \leq \epsilon$ is sufficiently small for γ in a finite generating set, then $z(\gamma) = v - \Phi(\gamma)v \in B^1(\Gamma, V)$ satisfies

$$||z(\gamma)|| \leq \epsilon ||v||$$

for γ in this same (finite) generating set. Since $H^1(\Gamma, V_\Phi) = 0$, there is a section Δ of the coboundary map $\partial^0 : \mathrm{Hom}_\Gamma(\mathbb{Z}[\Gamma], V) \to \mathrm{Hom}_\Gamma(I_F/I_F\overline{I}_R)$ satisfying $||\Delta(z)|| \leq K||z||$ for some positive constant $K > 0$. By choosing Ψ close enough to Φ so that ϵ satisfies $K\epsilon < 1/2$, gives

$$||\Delta(\partial^0(v))|| \leq K\epsilon ||v||$$

and so $v - \Delta(\partial^0(v))$ is a vector in $H^0(\Gamma, V_\Psi)$ with

$$||v|| - \Delta(\partial^0(v)) \geq 1/2||v||.$$

Since $H^0(\Gamma, V_\Phi) = 0$, $||v - \Delta(\partial^0(v))|| = 0$ and so $v = 0$. □

The proof of Theorem A now follows fairly straightforwardly from Lemma 13 as follows. Given an Anosov-like action $\phi \in \mathrm{Act}^1(\Gamma, M)$ and any other $\psi \in \mathrm{Act}^1(\Gamma, M)$, we define two actions of Γ on $C^0(M, M)$, Φ and Ψ by:

$$\begin{aligned} \Phi(\gamma)(f) &= \psi(\gamma) \circ f \circ \phi(\gamma^{-1}) \\ \Psi(\gamma)(f) &= \phi(\gamma) \circ f \circ \psi(\gamma^{-1}) \end{aligned}$$

for any $f \in C^0(M, M)$. It is easily verified that both of these are in $\mathrm{Act}^1(\Gamma, C^0(M, M))$ and that if ψ is C^1 close to ϕ in $\mathrm{Act}^1(\Gamma, M)$, then both Φ and Ψ are C^1 close in $\mathrm{Act}^1(\Gamma, C^0(M, M))$ to the adjoint action Ad_ϕ defined by:

$$\mathrm{Ad}_\phi(\gamma)(f) = \phi(\gamma) \circ f \circ \phi(\gamma^{-1})$$

Since Ad_ϕ has the identity $Id \in C^0(M, M)$ as a fixed point and ϕ is Anosov-like, it is easy to see that the linearization of Ad_ϕ at the identity is ϕ_*. As $C^0(M, M)$ is a smooth manifold modelled on the Banach space $D^0(M)$ (see [2]) and $H^1(\Gamma, D^0(M)) = 0$ it follows from [6] that there are $f, g \in C^0(M, M)$ so that

$$\begin{aligned} \psi(\gamma) \circ f &= f \circ \phi(\gamma) \\ \phi(\gamma) \circ g &= g \circ \psi(\gamma) \end{aligned}$$

for all $\gamma \in \Gamma$. Clearly a result of this is that

$$\begin{aligned} (f \circ g) \circ \psi &= \psi \circ (f \circ g) \\ (g \circ f) \circ \phi &= \phi \circ (g \circ f) \end{aligned}$$

The remaining task is to show that this implies that $f \circ g = g \circ f = Id$. Let $(\mathcal{U}, F)$ be a chart in $C^0(M, M)$ around the identity with $F : \mathcal{U} \to D^0(M)$ satisfying $F(Id) = 0$ and $\gamma_1, \ldots \gamma_n$ be a finite set of generators for Γ. Define the function $\overline{\Phi} : F(\mathcal{U}) \to D^0(M)^n$ via

$$\overline{\Phi}(v) = (v - F(\phi(\gamma_1) \circ F^{-1}(v) \circ \phi(\gamma_1^{-1})), \ldots, v - F(\phi(\gamma_n) \circ F^{-1}(v) \circ \phi(\gamma_n^{-1})))$$

with a similar definition of $\overline{\Psi}(v)$ replacing ϕ with ψ. By definition of both $\overline{\Phi}$ and $\overline{\Psi}$, $h \in \mathcal{U}$ satisfies $h \circ \phi = \phi \circ h$ if and only if $\overline{\Phi}(F(h)) = 0$ and similarly for $\overline{\Psi}$. Clearly, $h = Id$ is a solution of $\overline{\Phi}(F(h)) = 0$ and by the implicit function theorem this will be an isolated solution if and only if $\ker(d\overline{\Phi}(0)) = \{0\}$. Given an Anosov-like action $\phi \in \mathrm{Act}^1(\Gamma, M)$, by Lemma 13 there is a neighborhood $\mathcal{U}$ of ϕ_* in $R(\Gamma, L(D^0(M)))$ so that for any $\psi \in \mathrm{Act}^1(\Gamma, M)$ with $\psi_* \in \mathcal{U}$, ψ is also Anosov-like. A short computation shows that

$$d\overline{\Phi}(0)(v) = (v - \phi(\gamma_1)_*(v), \ldots, v - \phi(\gamma_n)_*(v))$$

for all $v \in D^0(M)$ which has kernel precisely $H^0(\Gamma, D^0(M)) = \{0\}$ by the Anosov-like hypothesis. □

5. Proof Of Theorem B

By the Anosov-like hypothesis there are continuous (linear) sections

$$\Delta_1 : Z^1(\Gamma_1, D^0(M)_{\phi_*}) \to D^0(M) \text{ and } \Delta_2 : Z^1(\Gamma_2, D^0(N)_{\psi_*}) \to D^0(N)$$

of the co-boundary operators

$$\partial_1 : D^0(M) \to Z^1(\Gamma_1, D^0(M)_{\phi_*}) \text{ and } \partial_2 : D^0(N) \to Z^1(\Gamma_2, D^0(N)_{\psi_*}).$$

Also let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be the projections onto the first and second factors respectively with the bundles $E_1 = \pi_1^* TM$ and $E_2 = \pi_2^* TN \subset T(M \times N)$ the pullbacks of the tangent bundles of M and N. By construction $E_1 \bigoplus E_2 = T(M \times N)$ and the direct sum decomposition is invariant under the action $\phi \times \psi$.

For a 1-cocycle $z_{\gamma_1 \times \gamma_2}(x, y)$ in $Z^1(\Gamma_1 \times \Gamma_2, D^0(M \times N))$, write $z = z_1 + z_2$ where the z_is are sections of E_1 and E_2 respectively. Both groups Γ_1 and Γ_2 act on $M \times N$ via the actions $\phi \times 1$ and $1 \times \psi$ respectively, these actions being the restrictions of the product action to the subgroups Γ_1 and Γ_2 of the product group. Restricting z to these subgroups and using the cocycle condition shows that:

$$z_{\gamma_1 \delta_1 \times 1}(x, y)_1 = z_{\gamma_1 \times 1}(x, y)_1 + \phi_{\gamma_1 *}(z_{\delta_1 \times 1}(x, y)_1) \text{ and}$$

$$z_{\gamma_1 \delta_1 \times 1}(x, y)_2 = z_{\gamma_1 \times 1}(x, y)_2 + z_{\delta_1 \times 1}(x, y)_2$$

Defining $v(x, y)$ by $v(x, y) = \Delta_1(z_1((., y))(x)$ shows that $z_1|_{\Gamma_1 \times 1}$ is cohomologous to 0 (i.e. $z_{\gamma_1 \times 1}(x, y)_1 = v(x, y) - (\phi_{\gamma_1} \times 1)_* v(x, y)$. Without loss of generality we cna therefore assume that $z_1 = 0$.

For z_2, the equation $[\gamma_1, \gamma_2] = 0$ for all $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ implies that

$$(1 - \psi_{1 \times \gamma_2 *}) z_{\gamma_1 \times 1}(x, y)_2 = 0$$

which in turn implies that $z_2 = 0$. By symmetry, we have $z|_{1 \times \Gamma_2} = 0$. As cocycles are determined by their values on generators this gives $H^1(\Gamma_1 \times \Gamma_2, D^0(M \times N)_{(\phi \times \psi)_*}) = 0$. □

6. Proof Of Theorem C

Since $\phi(\gamma)$ is Anosov for some $\gamma \in \Gamma$, $H^0(\Gamma_p, T_pM) = 0$ for all $p \in \mathrm{Per}(\phi)$. From our the hypotheses that Γ satisfies the SVC and that $H^0(\Gamma_p, T_pM) = 0$, for each $p \in \mathrm{Per}(\phi)$, there is a unique $v_p \in T_pM$ so that $v_p - d\phi_\gamma(p).v_p = z_\gamma(p)$ for all $\gamma \in \Gamma_p$. The function $p \to v_p$ defines a section of $T_{\mathrm{Per}(\phi)}M$ (that is not necessarily continuous) over $\mathrm{Per}(\phi)$. We first show that the vector field $V(p) = v_p$ over $\mathrm{Per}(\phi)$ satisfies $z(p) = V(p) - \phi(\gamma)_*(V)(p)$ for all $\gamma \in \Gamma$ and $p \in \mathrm{Per}(\phi)$. The next step is to show that the vector field V is continuous on $\mathrm{Per}(\phi)$. Having done this, we can extend V to a continuous tangent vector field $\overline{V}$ on M by continuity of V and the hypothesis that $\mathrm{Per}(\phi)$ is dense in M. Since the extended vector field $\overline{V}$ satisfies $z(p) = \overline{V}(p) - \phi(\gamma)_*(\overline{V})(p)$ for p in a dense set, it therefore satisfies the same equation for all $p \in M$.

Given a $p \in \mathrm{Per}(\phi)$ let $\tilde{\Gamma}_p = \cap_{\gamma\in\Gamma}\gamma \cdot \Gamma_p \cdot \gamma^{-1}$ be the normal core of Γ_p. Letting $\overline{z}(\gamma)(p) = z(p) - V(p) - \phi(\gamma)_*(V)(p)$ for any $\gamma \in \Gamma$, from the above $\overline{z}(\gamma)(p) = 0$ for $\gamma \in \Gamma_p$. Let $G = \Gamma/\tilde{\Gamma}_p$ and let $\{\tilde{g}\}$ be a set of coset representatives in Γ for $\{g \in G\}$. Given a $\gamma \in \Gamma$ there are $\tilde{\gamma}$ and $\tilde{\gamma} \in \tilde{\Gamma}_p$ so that $\gamma = \tilde{g} \cdot \tilde{\gamma} = \tilde{\gamma}' \cdot \tilde{g}$ where $g = \gamma\tilde{\Gamma}_p$. Computing $\overline{z}(\gamma)(p)$ using both of these and comparing the resulting expressions gives:

$$\begin{aligned}\overline{z}(\gamma)(p) &= \overline{z}(\tilde{g}\tilde{\gamma}) = \overline{z}(\tilde{g}) + \phi(\tilde{g})_*(\overline{z}(\tilde{\gamma}))(p)\\ &= \overline{z}(\tilde{g})\\ &= \overline{z}(\tilde{\gamma}'\tilde{g}) = \overline{z}(\tilde{\gamma}') + \phi(\tilde{\gamma}')_*(\overline{z}(\tilde{g}))\\ &= \phi(\tilde{\gamma}')_*(\overline{z}(\tilde{g}))\end{aligned}$$

since $\overline{z}(\tilde{\gamma})(p) = 0$. The above therefore gives us that $\overline{z}(\tilde{g}) \in H^0(\tilde{\Gamma}_p, T_pM) = 0$ since $\tilde{\Gamma}_p$ has finite index in Γ. Thus $\overline{z}(\gamma)(p) = 0$ for all $\gamma \in \Gamma$ and $p \in \mathrm{Per}(\phi)$.

Having shown that our tangent vector field is a solution for each $p \in \mathrm{Per}(\phi)$, we now show that it is continuous. Let $<\gamma_0>= \Gamma' \subset \Gamma$. By definition of γ_0, $\Gamma' \cong \mathbb{Z}$ and taking the restriction of $z(\gamma)$ to Γ', we get a $V' \in D^0(M)$ so that $z(\gamma_0^n) = V' - \phi(\gamma_0^n)_*(V')$. It is easily verified that for each $x \in \mathrm{Per}(\phi)$, $V(x) = V'(x)$ since $H^0(\Gamma', D^0(M)) = 0$ and so therefore $V'(x)$ is a continuous extension of $V(x)$ to all of M since $\mathrm{Per}(\phi)$ is dense. □

Note that Theorem C is not the strongest possible theorem that can be proven in this direction although it has the benefit of having a fairly succinct proof and also follows the same general logical structure of most of the proofs of infinitesimal rigidity in the literature. As most authors interested in local rigidity have needed some form of regularity to the conjugacies, the literature tends to focus on finding vanishing results for $H^1(\Gamma, D^r(M))$ rather than the topological case examined here.

7. Conclusion

We close by again remarking that the results here do not in general give a model for the dynamics of the action of the group Γ. The results generalize the stability properties of the class of Anosov diffeomorphisms to actions of more general groups (the class of finitely generated ones). Interesting directions for further research would be to try to relativize the results above and so generalize the notion of an Axiom A action to more general group actions.

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