

On Obstructions to Realizing Diagrams of Π -algebras

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Overview

Collaboration with [David Blanc](#) and [Jim Turner](#).

Π -alg, the category of Π -algebras, is the formal home for the (graded) homotopy groups of a (pointed) topological space, including the action of the primary homotopy operations.

Given $\Lambda \in \Pi\text{-alg}$, a [realization](#) of Λ refers to any space X with $\pi_* X \approx \Lambda$.

[Blanc-Dwyer-Goerss \(Topology, 2004\)](#) gave an obstruction theory for the existence of a realization for Λ using [André-Quillen cohomology](#). I will describe an extension to [diagrams in \$\Pi\$ -alg](#), together with some [spectral sequences](#) to compute the generalized obstructions.

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Motivation

Higher homotopy operations, such as [Toda brackets](#), may also be described as obstructions to finding a strictly commutative model for a homotopy commutative diagram.

After applying π_* to the homotopy commutative diagram, these higher homotopy operations may be viewed as obstructions to realizing a diagram in Π -alg. We are working toward understanding the relationship between higher homotopy operations and the k -invariants for our obstruction theory.

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After applying π_* to the homotopy commutative diagram, these higher homotopy operations may be viewed as obstructions to realizing a diagram in Π -alg. We are working toward [understanding the relationship](#) between higher homotopy operations and the k -invariants for our obstruction theory.

Related Work

Blanc-Chachólski are still working on the final details of a definition of **pointed higher homotopy operations in a model category**, improving on that of Blanc-Markl (Math Z., 2003).

Dwyer-Kan-Smith (JPAA, 1989) gave an obstruction theory for realizing diagrams in Π -alg with specified CW complexes as objects, via a theory I will call **\mathcal{SO} -cohomology**.

Baues-Blanc have also been working to identify **Baues-Wirsching cohomology**, André-Quillen cohomology, and **\mathcal{SO} -cohomology** in a way which preserves the relevant obstruction theories.

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Some Definitions

Let Π denote the full subcategory of the homotopy category of (pointed, connected) spaces consisting of **finite wedges of spheres** $\bigvee_{i=1}^k S^{n_i}$ with each $n_i > 0$ and $k > 0$.

$\Pi\text{-alg}$ will denote the category of product-preserving functors from Π^{op} into Set_* . (Note: the opposite category formally converts wedges to products).

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More Definitions

Π encodes primary homotopy operations like **Whitehead products** as well as compositions. For example, a homotopy class $h : S^n \rightarrow S^1 \vee S^n$ yields

$$\pi_1 X \times \pi_n X \approx [S^1 \vee S^n, X]_* \xrightarrow{h^*} [S^n, X]_* \approx \pi_n X.$$

All natural primary operations arise in this way for formal reasons.

Given $\Lambda \in \Pi\text{-alg}$, its loop Π -algebra, $\Omega\Lambda$, is defined by precomposition with the suspension functor. Thus $\Omega\Lambda(U) = \Lambda(\Sigma U)$ which implies $\pi_*(\Omega X) = \Omega\pi_* X$. Of course $\Omega^n \Lambda$ is defined similarly, and forms a “module” over Λ .

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Obstruction Theory

Theorem: A diagram Λ in Π -alg can be realized by a strictly commutative diagram of spaces if and only if an inductively defined sequence of **cohomology classes in $H_{\Lambda}^{n+2}(\Lambda; \Omega^n \Lambda)$ all vanish**. The different realizations (if any) are classified (up to homotopy) by elements of $H_{\Lambda}^{n+1}(\Lambda; \Omega^n \Lambda)$.

Unfortunately, this cohomology theory is André-Quillen cohomology of a diagram in Π -alg, which **had not been calculated** for any examples we could find. Thus, we sought computational tools.

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Arrow Case

The André-Quillen cohomology of individual Π -algebras had been computed in some cases and discussed many places.

We constructed a long exact sequence for an arrow in Π -alg. The pieces were the usual cohomology of the source and target together with another term involving cohomology of the source with coefficients coming from the target.

$$\cdots \rightarrow H^i(\Lambda; M) \rightarrow H^i(\Lambda_0; M_0) \oplus H^i(\Lambda_1; M_1) \rightarrow H^i(\Lambda_0; M_1) \rightarrow \cdots$$

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Two Approaches

The LES for an arrow generalizes to any **tree-like indexing category**. We sought spectral sequences to generalize to the case of a more complex finite index.

We followed two approaches to building “local-to-global spectral sequences” from a tower of fibrations:

- by filtering/truncating the indexing category, or
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Example Indexing Category

Truncating

$$1$$

$$2 \longrightarrow 1$$

$$\begin{array}{c} 3 \\ \downarrow \\ 2 \longrightarrow 1 \end{array}$$

$$\begin{array}{ccc} 4 & \longrightarrow & 3 \\ \downarrow & & \downarrow \\ 2 & \longrightarrow & 1 \end{array}$$

Composing

$$4 \longrightarrow 1$$

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First Spectral Sequence

The construction of the “truncation” spectral sequence relies upon an observation about certain **units of adjunction becoming fibrations** for Reedy model category structures.

Theorem: A “left-orderable” filtration of a directed index yields a **tower of fibrations**. The terms are mapping spaces between restrictions of diagrams to truncations of the index, and the inverse limit is the mapping space between the full diagrams.

As a consequence we have a first quadrant spectral sequence with E^1 -term given by a certain “relative cohomology” theory, with coefficients in the fiber of the unit of the truncation adjunction.

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Second Spectral Sequence

The key lemma for the second type of spectral sequence shows that for index $3 \xrightarrow{\varphi} 2 \xrightarrow{\psi} 1$ the difference map of the form

$$Y(\psi)_* - X(\varphi)^* : \text{map}(X_3, Y_2) \times \text{map}(X_2, Y_1) \rightarrow \text{map}(X_3, Y_1)$$

is a **fibration of simplicial abelian groups** for Y a fibrant abelian group object and X cofibrant (in a simplicial model category).

Theorem: Given an ordering of the objects of a finite indexing category, there is a first quadrant spectral sequence converging to André-Quillen cohomology of diagrams, with E^1 -term given in terms of “local cohomology at an object”.

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Summary

There is an obstruction theory for the realization of a diagram in Π -alg, with the relevant obstructions as classes in André-Quillen cohomology of diagrams.

There are computational tools which would allow one to detect these obstructions, at least in simple cases.

These obstructions are closely linked to other realization obstruction theories (Dwyer-Kan-Smith, Baues, dual cohomology questions).

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