# Finite generation of Tate cohomology and Freyd's generating hypothesis

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# The generating hypothesis

**The Generating Hypothesis of Peter Freyd (1965):**
The stable homotopy functor $\pi_*(-)$ is faithful on the category of finite spectra.

The GH if true reduces the study of finite spectra $X$ to the study of their homotopy groups $\pi_*(X)$ as modules over $\pi_*(S^0)$.

This problem is known to be notoriously hard. Ethan Devinatz proved a partial affirmative result using the machinery of nilpotence and periodicity theory.

# The generating hypothesis in derived categories

To gain some insight into the GH we study its analogues in stable homotopy categories.

**Theorem (Lockridge, 2005)**

*Let $R$ be a commutative ring. The GH holds in the derived category of $R$ (i.e., the homology functor $H_*(-)$ on category of perfect complexes is faithful) if and only if $R$ is von Neumann regular.*

Mark Hovey and Keir Lockridge have proved some further results in this direction.

# The stable module category

$G$ is a finite group and $k$ is a field of characteristic $p > 0$ which divides the order of $G$.

$f, g \colon M \to N$ are **homotopic** if their difference $f - g$ factors through a projective.

**The stable module category** – stmod($kG$)

- Objects: finite–dimensional $kG$–modules.
- Morphisms: homotopy classes of $kG$–linear maps.

**Tate cohomology functor:**

$$\begin{aligned} \widehat{\mathrm{H}}^*(G, -) : \quad \mathrm{stmod}(kG) &\longrightarrow \text{graded vector spaces over } k. \\ M &\longmapsto \widehat{\mathrm{H}}^*(G, M) \end{aligned}$$

stmod($kG$) is a **triangulated category**:

1. $\Omega(M) := \ker(P_M \twoheadrightarrow M)$
2. exact triangles are given by the short exact sequences of $kG$–modules.

**Fact:** $\widehat{\mathrm{H}}^i(G, M) \cong \underline{\mathrm{Hom}}_{kG}(\Omega^i k, M)$ for all $i$.

Thick subcategory $\mathrm{thick}_G(k)$ generated by $k$ is the smallest subcategory of stmod($kG$) that is closed under cofibre sequences and retractions.

**Problem:** For which finite groups $G$, is the Tate cohomology functor $\widehat{\mathrm{H}}^*(G, -)$ faithful on the thick subcategory generated by $k$? In other words, when does the GH hold in $\mathrm{thick}_G(k)$?

# The GH in the stable module category

**Theorem (Carlson–C.–Mináč, 2007)**

*The generating hypothesis holds for kG if and only if the Sylow p–subgroup of G is either $C_2$ or $C_3$.*

This builds on some previous work with Dave Benson and Dan Christensen.

Proof uses Block theory, structure of the principal block for cyclic Sylow $p$–subgroups, Auslander–Reiten sequences, varieties of modules, some general constructions which make use of the triangulated structure of the stable module category.

**Question:** Which representation theoretic property distinguishes groups whose Sylow $p$–subgroup is either $C_2$ or $C_3$ amongst finite groups?

**Answer:** These are the only groups with the property that every $kG$–module in the thick subcategory generated by $k$ is a direct sum of suspensions of $k$.

# Finite generation of Tate cohomology

In our work on the GH we were led naturally to the following question.

**Question:** Let $M$ be a finitely–generated $kG$–module. Is it true that $\widehat{\mathrm{H}}^*(G, M)$ is finitely generated as a graded module over $\widehat{\mathrm{H}}^*(G, k)$?

Theorem (Evens–Venkov)

*Let $M$ be a finitely–generated kG–module. Then $H^*(G, M)$ is finitely generated as a graded module over $H^*(G, k)$.*

The answer to the above question is a *resounding NO!!*

# Counter-examples to the GH

**Theorem (Carlson–C.–Mináč, 2007)**

*Let G be a group with non–periodic cohomology. Let $\zeta$ be an element in the Tate cohomology ring of G such that*

$$Dim\ \ Im[\zeta\colon \widehat{\mathrm{H}}^*(G,k) \longrightarrow \widehat{\mathrm{H}}^*(G,k)] < \infty$$

*Then the Tate cohomology of the module M represented by $\zeta$ is finitely generated over $H^*(G,k)$.*

If $\zeta$ belongs to $\widehat{\mathrm{H}}^{n+1}(G,k)$, then there exits a module $M$ which sits in:

$$\zeta\colon 0 \longrightarrow k \longrightarrow M \longrightarrow \Omega^n k \longrightarrow 0$$

# Groups acting freely on spheres

Theorem (Carlson–C.–Mináč, 2007)

$\widehat{\mathrm{H}}^*(G, M)$ *is finitely generated over* $\widehat{\mathrm{H}}^*(G, k)$ *for all k and all finitely generated kG–modules M if and only if G acts freely on a finite complex with has the homotopy type of a sphere.*

**Strategy:**

- (Swan) $G$ acts freely on a finite complex $X \simeq S^n$ for some $n$ if and only if every abelian subgroup of $G$ is cyclic.
- Every abelian group of $G$ is cyclic if and only if the $p$–rank of $G$ is one for all primes $p$.

- If the $p$–rank of $G$ is one, then $\widehat{\mathrm{H}}^*(G, k)$ is periodic whenever characteristic of $k$ is $p$. So the result is clear.
- If the $p$–rank of $G$ is at least two, then we show that $\widehat{\mathrm{H}}^*(G, \mathrm{End}_k(L))$ is not finitely generated as a $\widehat{\mathrm{H}}^*(G, k)$–module for all non–projective $kG$–periodic modules $L$.

# Our Conjecture

For an indecomposable non–projective $kG$–module $M$, $\widehat{\mathrm{H}}^*(G, M)$ is finitely generated over $\widehat{\mathrm{H}}^*(G, k)$ if and only if $V_G(M) = V_G(k)$.

$$V_G(M) := \mathrm{Var}\,[\mathrm{Ker}\colon\ \mathrm{Ext}^*_{kG}(k, k) \longrightarrow \mathrm{Ext}^*_{kG}(M, M)]$$

# Affirmative result

The following result gives some support to our conjecture.

**Theorem (Carlson–C.–Mináč, 2007)**

*Let $G$ be a group with the property that product of any two elements in negative cohomology is trivial. If $V_G(M) \subseteq V_G(\zeta)$ for some regular element $\zeta$ in the cohomology ring, then $\widehat{\mathrm{H}}^*(G, M)$ is not finitely generated over $\widehat{\mathrm{H}}^*(G, k)$.*

*Dhanyavad*

*Thank You*