The splitting of $b\mathrm{o} \wedge \mathit{tmf}$

Scott M. Bailey

NORTHWESTERN
UNIVERSITY

November 4, 2007

Some motivation:

- Let $\mathcal{A}$ be the mod-2 Steenrod algebra.
- $\mathcal{A}(n)$ is the subalgebra generated by $\{\mathrm{Sq}^1, \ldots, \mathrm{Sq}^{2^n}\}$.
- The inclusions $\mathbb{F}_2 \to \mathcal{A}(0) \to \mathcal{A}(1) \to \mathcal{A}(2) \to \cdots$ give rise to morphisms

$$\mathcal{A} \otimes_{\mathbb{F}_2} \mathbb{F}_2 \to \mathcal{A} \otimes_{\mathcal{A}(0)} \mathbb{F}_2 \to \mathcal{A} \otimes_{\mathcal{A}(1)} \mathbb{F}_2 \to \mathcal{A} \otimes_{\mathcal{A}(2)} \mathbb{F}_2 \to \cdots$$

$$H^* H\mathbb{F}_2 \to H^* H\mathbb{Z}_2 \to H^* bo \to H^* tmf$$

$$H\mathbb{F}_2 \leftarrow H\mathbb{Z}_2 \leftarrow bo \leftarrow tmf$$

$$H_* H\mathbb{F}_2 \leftarrow H_* H\mathbb{Z}_2 \leftarrow H_* bo \leftarrow H_* tmf$$

$$\mathbb{F}_2[\zeta_1, \ldots] \leftarrow \mathbb{F}_2[\zeta_1^2, \zeta_2, \ldots] \leftarrow \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \ldots] \leftarrow \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \ldots]$$

bo

- Mahowald (1981) showed $bo \wedge bo$ splits.
- Carlsson (1976) calculated $[bo, bo]$

$bo \leftarrow tmf$

- Bailey (2007) shows $bo \wedge tmf$ splits
- Bailey (2007) calculates $[tmf, bo]$

tmf

We would like to:

- Understand $tmf \wedge tmf$ - splits, but much more complicated.
- Understand $[tmf, tmf]$.

The splitting of $bo \wedge tmf$ should also give insight to the splitting of the Tate spectrum of tmf.

Integral Brown-Gitler spectra, $B_1(n)$

Defining Properties of $B_1(n)$

(a) $H^*B_1(n) = M_1(n)$

$$M_1(n) = \mathcal{A}/\mathcal{A}(\mathrm{Sq}^1, \chi\,\mathrm{Sq}^k \mid k > 2n)$$

(b) $B_1(n) \to K\mathbb{Z}_2$ induces a surjection

$$B_1(n)_q(X) \to H_q(X, \mathbb{Z}_2) \qquad q \leq 4n+1$$

for all CW-complexes X.

Multiplication of $B_1(n)$

There are maps $B_1(m) \wedge B_1(n) \to B_1(n+m)$ such that the induced map in homology is onto if $n = 2^i$ and $m < 2^i$.

Statement of Main Theorem

Theorem (Bailey).

There is a splitting of $b\mathrm{o}$-module spectra

$$b\mathrm{o} \wedge tmf \simeq \bigvee_{0 \leq j \leq i} \Sigma^{8i+4j} b\mathrm{o} \wedge B_1(j)$$

This theorem can be restated in a form analagous to the Davis's splitting of $b\mathrm{o} \wedge MO\langle 8 \rangle$.

Cohomological Splitting as left $\mathcal{A}(1)$-modules

Left $\mathcal{A}(1)$-module structure of H^*tmf

$$H^*tmf \cong \bigoplus_{0\leq j\leq i} \Sigma^{8i+4j} H^* B_1(j)$$

Note: Since $H^*tmf \cong \bigoplus_{0\leq j\leq i} \Sigma^{8i+4j} H^* B_1(j)$ as an $\mathcal{A}(1)$-module, $H^*(bo \wedge tmf) \cong \bigoplus_{0\leq j\leq i} \Sigma^{8i+4j} H^*(bo \wedge B_1(j))$ as an $\mathcal{A}$-module.

$\mathcal{A}(2)$-module structure of $H_* tmf$

- Simplify calculations by dualizing.
- $H_* tmf \cong \mathbb{F}_2[\zeta_1^8, \zeta_2^4, \zeta_3^2, \zeta_4, \ldots]$ where $|\zeta_i| = 2^i - 1$.
- Define a new weight on the generators by

$$\omega(\zeta_i) = 2^{i-1} \qquad \omega(\zeta_1^{i_1} \cdots \zeta_n^{i_n}) = \sum_{\ell=1}^{n} i_\ell 2^{\ell-1}$$

- Let $N_{8n}^{tmf} = \{x \in H_* tmf \,|\, \omega(x) = 8n\}$
- The right action of $\mathcal{A}(2)$ on $H_* tmf$ is weight preserving, hence there is an isomorphism of right $\mathcal{A}(2)$ modules:

Right $\mathcal{A}(2)$-module structure of $H_* tmf$

$$H_* tmf \cong \bigoplus_{n \geq 0} N_{8n}^{tmf}$$

Splitting of N_{8n}^{tmf}

- $H_* b\mathrm{o} \cong \mathbb{F}_2[\zeta_1^4, \zeta_2^2, \zeta_3, \zeta_4, \cdots]$.
- Let $N_{4n}^{b\mathrm{o}} = \{x \in H_* b\mathrm{o} \,|\, \omega(x) = 4n\}$.
- The right action of $\mathcal{A}(1)$ on $H_* b\mathrm{o}$ is weight preserving, hence there is an isomorphism of right $\mathcal{A}(1)$-modules

Right $\mathcal{A}(1)$-module structure of $H_* b\mathrm{o}$

$$H_* b\mathrm{o} \cong \bigoplus_{n \geq 0} N_{4n}^{b\mathrm{o}}$$

- We can further split the summands of $H_* tmf$ as right $\mathcal{A}(1)$-modules

$$\Sigma^{-8n} N_{8n}^{tmf} \cong \bigoplus_{j=0}^{n} N_{4j}^{b\mathrm{o}}$$

$\mathcal{A}(1)$-module structure of $H^* tmf$

Right $\mathcal{A}(1)$-module structure of $H_* tmf$

$$H_* tmf \cong \bigoplus_{0 \leq j \leq i} \Sigma^{8i} N^{bo}_{4j}$$

- Mahowald showed

$$\left(N^{bo}_{4j} \right)^* \cong \Sigma^{4j} M_1(j)$$

- After dualizing, there is an isomorphism of left $\mathcal{A}(1)$-modules

Left $\mathcal{A}(1)$-module structure of $H^* tmf$

$$H^* tmf \cong \bigoplus_{0 \leq j \leq i} \Sigma^{8i+4j} M_1(j) \cong \bigoplus_{0 \leq j \leq i} \Sigma^{8i+4j} H^* B_1(j)$$

Sketch Proof of Main Theorem (via induction)

- Define $\Omega^n = \bigvee_{j=0}^{n} \bigvee_{i=j}^{\infty} \Sigma^{8i+4j} B_1(j)$ to be the wedge containing the first $n+1$ integral Brown-Gitler spectra and their 8-fold suspensions.
- Assume there is a map $bo \wedge \Omega^{2^i-1} \to bo \wedge tmf$ which induces a stable $\mathcal{A}$-isomorphism through dimension $12(2^i)-1$.
- Construct a map $\Sigma^{12(2^i)} bo \wedge B_1(2^i) \to bo \wedge tmf$ inducing an injection in homology to extend to a map

$$bo \wedge \Omega^{2^i} \to bo \wedge tmf$$

which is a stable $\mathcal{A}$-isomorphism through dimension $12(2^i)+11$.

Multiplication of $B_1(n)$

There are maps $B_1(m) \wedge B_1(n) \to B_1(n+m)$ such that the induced map in homology is onto if $n = 2^i$ and $m < 2^i$.

Extension via multiplication

- Suppose there are maps

  $$g_\ell : \Sigma^{12\ell} B_1(\ell) \to b\mathrm{o} \wedge tmf$$

  so that

  $$\Sigma^{12\ell} b\mathrm{o} \wedge B_1(\ell) \to b\mathrm{o} \wedge tmf$$

  induces an injection on homology for all $0 \leq \ell \leq 2^i$.
- Since there are no maps from $H\mathbb{F}_2 \to b\mathrm{o} \wedge tmf$, for $n < 2^i$ there is an extension

$$\begin{tikzcd}
b\mathrm{o} \wedge \Sigma^{12n} B_1(n) \wedge \Sigma^{12(2^i)} B_1(2^i) \arrow[r] \arrow[d, "g_n \wedge g_{2^i}"'] & (b\mathrm{o} \wedge \Sigma^{12(n+2^i)} B_1(n+2^i)) \vee K \arrow[d, dashed, "g_{n+2^i}"] \\
(b\mathrm{o} \wedge tmf) \wedge (b\mathrm{o} \wedge tmf) \arrow[r] & b\mathrm{o} \wedge tmf
\end{tikzcd}$$

where K is a wedge of suspensions of $H\mathbb{F}_2$'s.

"Bootstrapping our way up!"

- Use multiplication of integral Brown-Gitler spectra and the ring structure of $b\mathrm{o} \wedge tmf$ to extend this map to

$$b\mathrm{o} \wedge \Omega^{2^{i+1}-1} \longrightarrow b\mathrm{o} \wedge tmf$$

which is a stable $\mathcal{A}$-isomorphism through dimension $12(2^{i+1}) - 1$.

Key Step

Construct maps

$$\Sigma^{12(2^i)} b\mathrm{o} \wedge B_1(2^i) \longrightarrow b\mathrm{o} \wedge tmf$$

which induce an injection on homology groups.

- The multiplication map $B_1(2^{i-1}) \wedge B_1(2^{i-1}) \longrightarrow B_1(2^i)$ is not onto in homology.
- To complete the construction, appeal to the E_∞ structure of $b\mathrm{o} \wedge tmf$, the quadratic construction on $B_1(2^{i-1})$ and methods and calculations of Davis.

Define $h_{i-1} = \mathrm{m}(g_{2^{i-1}} \wedge g_{2^{i-1}})$ where m is the multiplication map on $b\mathrm{o} \wedge tmf$.
Let $D_2(X) = S^1 \ltimes_T (X \wedge X)$. Since $b\mathrm{o} \wedge tmf$ is an E_∞-ring spectrum, there is a factorization of h_{i-1}:

$$\begin{tikzcd}
b\mathrm{o} \wedge \Sigma^{2^{i+4}-5} M \wedge B_1(1) \arrow[d, "\delta"] & \Sigma^{12(2^i)} b\mathrm{o} \wedge D_2\left(B_1(2^{i-1})\right) \arrow[d] \\
\Sigma^{12(2^i)} b\mathrm{o} \wedge B_1(2^{i-1}) \wedge B_1(2^{i-1}) \arrow[r, "h_{i-1}"] \arrow[ur, "j"] \arrow[d] & b\mathrm{o} \wedge tmf \\
\Sigma^{12(2^i)} b\mathrm{o} \wedge B_1(2^i) \arrow[ur, dashed, "g_i"'] &
\end{tikzcd}$$

Davis calculates the induced map on homotopy j_*.

Relation to Adams covers

- If X is a spectrum, we can define the ℓ-th Adams cover of X as the spectrum satisfying the universal property: If there are ℓ maps which are zero in homology then there is a factorization

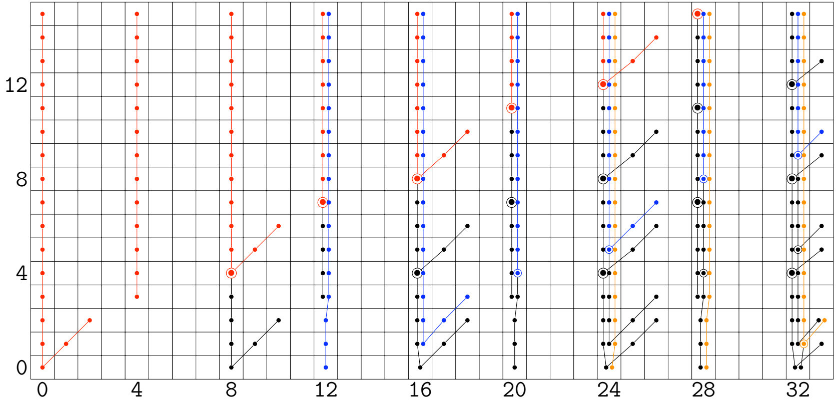
$$\begin{tikzcd}
Y_\ell \arrow[r] \arrow[rrrd, dashed] & \cdots \arrow[r] & Y_0 \arrow[r] & X \\
 & & & X^\ell \arrow[u]
\end{tikzcd}$$

- X^ℓ is unique up to $H\mathbb{Z}_2$'s.
- $\mathrm{Ext}_{\mathcal{A}}^{s,t-s}(H^*X^\ell, \mathbb{F}_2) \cong \mathrm{Ext}_{\mathcal{A}}^{s+\ell,t-s}(H^*X, \mathbb{F}_2)$ for $s > 0$.

Proposition (Mahowald).

$$b\mathrm{o} \wedge B_1(n) \simeq K \vee \begin{cases} b\mathrm{o}^{2n-\alpha(n)} & \text{if } n \text{ even} \\ b\mathrm{sp}^{2n-1-\alpha(n)} & \text{if } n \text{ odd} \end{cases}$$

$\mathrm{Ext}_{\mathcal{A}}^{s,t-s}(H^*(bo \wedge tmf), \mathbb{F}_2) \Rightarrow \pi_{t-s}(bo \wedge tmf)$



⋆⋆There can be no differentials for degree and naturality reasons, and theorem of Margolis.