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★**Lectures on Morse homology.**

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This book provides a treatment of finite-dimensional Morse theory and its associated chain complex, pitched at a level appropriate to early-stage graduate students. The fact that a Morse-Smale function on a Riemannian manifold gives rise to a chain complex whose homology is the singular homology of the manifold has been realized by a number of people in a number of different ways [see, e.g., R. Thom, C. R. Acad. Sci. Paris **228** (1949), 973–975; MR0029160 (10,558b); J. Milnor, *Lectures on the h-cobordism theorem*, Princeton Univ. Press, Princeton, N.J., 1965; MR0190942 (32 #8352); E. Witten, J. Differential Geom. **17** (1982), no. 4, 661–692 (1983); MR0683171 (84b:58111)], so the authors of such a book have a wide array of choices as to how to approach the subject. One approach which uses gluing and Fredholm theory is carried out in the text [M. Schwarz, *Morse homology*, Progr. Math., 111, Birkhäuser, Basel, 1993; MR1239174 (95a:58022)] and is similar in spirit to the methods of Floer theory, which provides an infinite-dimensional version of Morse homology under certain circumstances. The book under review takes a markedly different approach, viewing Morse homology more through the lens of classical topology and dynamical systems, and ultimately proving the main result using the theory of the Conley index, as was originally done in [D. A. Salamon, Bull. London Math. Soc. **22** (1990), no. 2, 113–140; MR1045282 (92a:58028)].

The book begins with a chapter which reviews some notions from algebraic topology, in particular introducing the notion of a CW-complex and proving that its cellular homology is isomorphic to its singular homology; one imagines that this material will be familiar to most readers of the book. There follows an introduction to the rudiments of Morse theory, in which the Morse inequalities are established and Morse-Bott theory is briefly discussed (without proofs in the latter case). The next chapter is devoted to a careful proof of the Stable/Unstable Manifold theorem, which asserts that if p is a critical point of a Morse function $f\colon M \to \mathbb{R}$ then the tangent space at p splits as $T_pM = T_p^sM \oplus T_p^uM$ with the stable (resp. unstable) manifold $W^s(p)$ (resp. $W^u(p)$) of p for the gradient flow of f appearing as the image of an embedding of T_p^sM (resp. T_p^uM) into M. The key analytical ingredient in the proof is an argument due to Irwin which uses the Lipschitz inverse function theorem. After this, the authors review some basic notions from differential topology such as transversality, and show that the set of Morse functions is open and dense in the smooth topology. Having proven these basic results, they can then proceed to the dynamical systems theory which underlies the main result of the book. One of the more important technical results is the so-called λ-lemma which was announced by Smale in 1960 and proven in [J. Palis, Topology **8** (1968), 385–404; MR0246316 (39 #7620)] and whose corollaries give crucial information about the closures of the sets $W^s(p) \cap W^u(q)$. This finally sets the stage for the definition of the Morse homology boundary operator and the proof, using Conley index theory, that its homology coincides with the cellular homology of the CW complex associated to the Morse function. Having proven the main result, the authors then give a pretty account of one detailed example of how the theory works: they describe the Morse complex associated to the Grassmannian $G_{n,n+k}(\mathbb{C})$ by embedding it as an adjoint orbit in the Lie algebra $\mathfrak{u}(n+k)$ and describing explicitly the critical points of a Morse function obtained by taking the inner product with an element of $\mathfrak{u}(n+k)$ having distinct eigenvalues. Finally the book closes with a brief account of symplectic, Lagrangian, and instanton Floer homologies.

Throughout, the authors take pains to make the material accessible, and while some proofs of standard results are skipped extensive references are provided. A number of readers will likely find the reviews of algebraic and differential topology unnecessary; guidance for them as to what to skip is provided in the introduction. Many well-drawn figures are provided to clarify the text, and there are over 200 exercises, with hints for some of them in the back.

As far as I know, this is the first time that this particular approach to Morse homology has appeared in book form. While the viewpoint taken in Schwarz's text [op. cit.] is important especially for those who are interested in Floer theory, this reviewer's experience would suggest that the Floer theory-inspired approach to Morse homology is in no danger of being underpublicized. As such, Banyaga and Hurtubise's book provides a valuable service by introducing young mathematicians to a circle of ideas which in recent years has perhaps received less attention than it deserves. *Michael J. Usher* (Princeton, NJ)